Kuwait University	**Math 101**	**Date:**	**June 21, 2009**
Dept. of Math. & Comp. Sci.	**Final Exam**	**Duration:**	**120 minutes**

Calculators, mobile phones, pagers and all other mobile communication equipments are not allowed

Answer the following questions:

1. Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to 1}(f(x)+g(x)) = 2$. If $f(x) = 3x + \frac{5}{x}$ and $\lim_{x\to 1} g(x)$ exists, find $\lim_{x\to 1} g(x)$. (4 pts.)

2. Find the values of the constant A such that
$$f(x) = \begin{cases} A(x^3 - 5x), & \text{if } x < 0, \\ \sin(x), & \text{if } x \geq 0 \end{cases}$$
is continuous at $x = 0$. (4 pts.)

3. Show that the curves $f(x) = x^2$ and $g(x) = -x^2 + 4x - 2$ have the same tangent line at their point of intersection. (4 pts.)

4. Let x and y be two real numbers whose sum is 20. Find the maximum value of their product. (4 pts.)

5. (a) Evaluate $\int x\sqrt{(x^2+7)^3}\, dx$. (2 pts.)

   (b) Can you evaluate $\int_0^1 \frac{2x^2 - x - 1}{2x+1}\, dx$? If yes, justify your answer and find the value of the integral. (2 pts.)

6. Show that $\int \sin^3(x)\, dx = -\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + C.$ (4 pts.)

7. Show that on any $[a, b]$: $\int_a^b (\cos(x) - 1)\, dx < \int_a^b (\sin(x) + 2)\, dx.$ (4 pts.)

8. Let $f(x) = \int_0^x (t^2 - t + 11)^{20}\, dt + \int_0^1 (x^2 - x + 11)^{20}\, dx.$ Show that $f(x)$ is increasing. (4 pts.)

9. Find the area between the curves $x - y^2 = -4$ and $x + y^2 = 4$. (4 pts.)

10. Set up an integral for the volume of the solid generated by revolving the region bounded by $y = x^2$ and $y = 1$ about:

    (a) the line $y = -1$. (2 pts.)

    (b) the line $x = 5$. (2 pts.)

1. $\lim_{x\to 1}(f(x)+g(x)) = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x) = 2$. Since $\lim_{x\to 1} f(x) = 3x + \frac{5}{x} = 8$. $\lim_{x\to 1} g(x) = 2 - 8 = -6$.

2. $f(0) = 0$, $\lim_{x\to 0^+} \sin(x) = 0$ and $\lim_{x\to 0^-} A(x^3 - 5x) = 0,\ \forall\ A \in \Re$.

3. Set $f(x) = g(x)$. That is, $x^2 = -x^2 + 4x - 2$. Hence, $2x^2 - 4x + 2 = 2(x-1)^2 = 0$: $x = 1$. $f'(1) = g'(1) = 2$: therefore, the two curves are tangent at $(1,1)$.

4. $x + y = 20$, thus $y = 20 - x,\ \forall\, x \in \Re$. Let $P = xy = x(20-x) = -x^2 + 20x$. $P' = -2x + 20 = -2(x-10)$. Critical number $x = 10$. $P''(x) = -2 < 0$: max value at $x = 10$: $P(10) = 100$ (max value).

5. Evaluate the integrals:

   (a) Let $u = x^2 + 7$, thus $du = 2x dx$. $\int x\sqrt{(x^2+7)^3}\, dx = \int \frac{1}{2} u^{3/2}\, du = \frac{u^{5/2}}{5} + C = \frac{(x^2+7)^{5/2}}{5} + C$.

   (b) Yes (discontinuous at $-\frac{1}{2} \notin [0,1]$). $\int_0^1 \frac{2x^2 - x - 1}{2x+1}\, dx = \int_0^1 \frac{(2x+1)(x-1)}{2x+1}\, dx = \frac{x^2}{2} - x]_0^1 = \frac{-1}{2}$.

6. $\frac{d}{dx}\left(-\frac{1}{3}\sin^2(x)\cos(x) - \frac{2}{3}\cos(x) + C\right) = \sin^3(x)$.

7. $\forall\ x, \cos(x) \leq 1$ *i.e.* $\cos(x) - 1 \leq 0$ and $\sin(x) \geq -1$ *i.e.* $\sin(x) + 1 \geq 0$: $\sin(x) + 2 > 0$. Thus, $\forall\ x$, $\cos(x) - 1 < \sin(x) + 2$,. Hence, $\int_a^b (\cos(x) - 1)\, dx < \int_a^b (\sin(x) + 2)\, dx$.

8. $f'(x) = (x^2 - x + 11)^{20} > 0, \forall\ x$. Thus, f is $\uparrow$.

9. $x = f(y) = y^2 - 4$ and $x = g(y) = 4 - y^2$. Set $f(y) = g(y)$: $2y^2 - 8 = 0$. Thus, $y = \pm 2$. $A = \int_{-2}^{2} \left((4 - y^2) - (y^2 - 4)\right) dy = \int_{-2}^{2} \left(-2y^2 + 8\right) dy = \frac{-2y^3}{3} + 8y]_{-2}^{2} = 64/3$.

10. Set $x^2 = 1$: $x = \pm 1$.

    (a) $V = \int_{-1}^{1} \pi\left((1+1)^2 - (x^2+1)^2\right) dx$; OR : $V = \int_0^1 2\pi(y+1)(2\sqrt{y})dy$.

    (b) $V = \int_{-1}^{1} 2\pi\left((5-x)(1-x^2)dx\right.$; OR : $V = \int_0^1 \pi\left((5+\sqrt{y})^2 - (5-\sqrt{y})^2\right) dy$.